**References**

[1] Chase, D. M., "Mean Velocity Profile of a Thick Turbulent Boundary along a Circular Cylinder." *AIAA Journal*, Vol. 10, No. 7, July 1972, pp. 849–850.

[2] Bradshaw, P. and Patel, V. C., "Comment on Mean Velocity Profile of a Thick Turbulent Boundary Layer along a Circular Cylinder." *AIAA Journal*, Vol. 11, No. 6, June 1973, pp. 893–894.

[3] Rao, G. N. V., "The Law of the Wall in a Thick Axisymmetric Turbulent Boundary Layer." *Journal of Applied Mechanics*, Vol. 34, 1967, pp. 237–238.

[4] Townsend, A. A., "Equilibrium Layers and Wall Turbulence." *Journal of Fluid Mechanics*, Vol. 11, Pt. 1, 1961, pp. 97–120.

# Comment on "Comparison of Linear and Riccati Equations Used to Solve Optimal Control Problems"

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IN a recent paper,[1] Tapley and Williamson discussed the application of various linear and Riccati methods to the computational solution of linearized versions of the two-point boundary-value problems (TPBVP's) arising in optimal control theory. Part of their conclusion, for the particular control problem considered, was that the Riccati methods "appear to be very 'unstable' from the standpoint of numerical integration techniques." This conclusion was apparently based on the computational experience that frequently "values for the exponents of the Riccati variables become larger than the computer can handle." The present authors believe this "blow-up" phenomenon is probably not numerical in nature, but rather is an indication of an actual singularity in the Riccati variables. In any event, such singularities are well-known[2,3] to occur sometimes in Riccati methods applied to optimal control problems, and consequently the following comments have relevancy.

The main purpose of this Comment is to point out the existence of known methods for continuing Riccati variables past a singularity. One such method is described briefly below, in order to illuminate this point. It is, of course, true that such methods cannot be based entirely upon step-by-step numerical integration schemes. The method described below is based upon the "recursive equations" (also known as "addition formulas") as discussed, for example, by Denman.[4] These relations are well-known within the field of invariant imbedding, and are essentially the same as the "generalized trigonometric identities" recently used by Allen and Wing[5–7] for the purpose of continuing Riccati variables past singularities. Yet another technique for accomplishing this has recently been discussed by Scott[8] and by Casti, Kalaba, and Scott.[9] Some recently published works[2,3] suggest that none of these methods are well-known to workers in control theory.

For convenience, the following discussion is referred to the approach called the "forward Riccati method" by Tapley and Williamson. The notation and terminology of Ref. 1 will be adopted as nearly as feasible. The primary objective of the Riccati method is, in an obvious extension of the notation of Ref. 1, to find $W(t_f, t_o)$ and $S(t_f, t_o)$, where $t_f$ and $t_o$ are given subject to $t_f > t_o$, and $W(t, t_o)$, $S(t, t_o)$ are $n \times n$ matrix functions determined by the initial-value problem

$$\dot{W} = A_{11}W + A_{12} - W(A_{21}W + A_{22}) \tag{1}$$

$$W(t_o, t_o) = 0 \tag{2}$$

$$\dot{S} = (A_{21}W + A_{22})S \tag{3}$$

$$S(t_o, t_o) = I \tag{4}$$

If $W(t, t_o)$ is finite for $t_o \leqq t \leqq t_f$, this poses no computational difficulty, at least in principle. From the identity $W = \Phi_1\Phi_2^{-1}$, it follows that this condition is equivalent to invertibility of $\Phi_2(t, t_o)$ for $0 \leqq t \leqq t_f$, or, equivalently, to the problem

$$\delta\dot{x} = A_{11}\delta x + A_{12}\delta\lambda \tag{5}$$

$$\delta\dot{\lambda} = A_{21}\delta x + A_{22}\delta\lambda \tag{6}$$

$$\delta x(t_o) = \delta\lambda(\tau) = 0 \tag{7}$$

having only the trivial solution for $0 \leqq \tau \leqq t_f$. For any given nominal trajectory, or even for an optimal trajectory, there appears to be no a priori reason to expect these conditions to hold. Of course, if Eqs. (5–7) have a nontrivial solution for $\tau$ near the true value of $t_f$, then the original nonlinear TPBVP is computationally unstable, which one can possibly expect (or hope) from physical considerations not to be the case. For guessed values of $t_f$, even this consideration is invalid. It would be of interest to consider the relationship between this possible source of computational instability and the necessity for the under-relaxation scheme for $h$ as described by Tapley and Williamson.

Let $T(t, t_o)$, $P(t, t_o)$, and $Q(t, t_o)$ be the $n \times n$ matrix functions defined by the initial-value system

$$\dot{T} = -T(A_{21}W + A_{22}) \tag{8}$$

$$T(t_o, t_o) = I \tag{9}$$

$$\dot{P} = [A_{11} - WA_{21}]P \tag{10}$$

$$P(t_o, t_o) = I \tag{11}$$

$$\dot{Q} = -TA_{21}P \tag{12}$$

$$Q(t_o, t_o) = 0 \tag{13}$$

Note that $S(t, t_o) = T(t, t_o)^{-1}$. For arbitrary $\tau$, let $W(t, \tau)$ $T(t, \tau)$, $P(t, \tau)$, and $Q(t, \tau)$ be defined by Eqs. (1, 2, and 8–13), with $t_o \to \tau$. If $t_o < t_1 < t_2$, the addition formulas

$$W(t_2, t_o) = W(t_2, t_1 + P(t_2, t_1)[I - W(t_1, t_o)Q(t_2, t_1)]^{-1} \times W(t_1, t_o)T(t_2, t_1) \tag{14}$$

$$T(t_2, t_o) = T(t_1, t_o)[I - Q(t_2, t_1)W(t_1, t_o)]^{-1}T(t_2, t_1) \tag{15}$$

follow from the semigroup property of the fundamental matrix for linear systems. If $t_1 - t_o$ and $t_2 - t_1$ are sufficiently small so that $W(t, t_o)$ and $W(t, t_1)$, respectively, are finite for $t_o \leqq t \leqq t_1$ and $t_1 \leqq t \leqq t_2$, and $I - W(t_1, t_o)Q(t_2, t_1)$ is invertible, then the quantities on the right-hand side of Eqs. (14) and (15) can be obtained by numerical integration of Eqs. (1, 2, and 8–13). With $W(t_2, t_o)$ and $T(t_2, t_o)$ known, these equations can then be integrated from $t = t_2$ to some $t = t_3$ over which $W$ is finite and which furthermore is such that $I - W(t_2, t_o) \cdot Q(t_3, t_2)$ is invertible. The addition formulas then give $W(t_3, t_o)$ and $T(t_3, t_o)$. After enough repetitions, some $t_n = t_f$ is reached, $W(t_f, t_o)$ is known, and $S(t_f, t_o) = T(t_f, t_o)^{-1}$ can be computed.

In order to illustrate further the possibility of continuing Riccati variables across singularities, consider the equation

$$\dot{R} = R + R^2 \sin t \tag{16}$$

which has the solution

$$R(t, t_o) = \frac{2\exp(t - t_o)R(t_o, t_o)}{2 - \exp(t - t_o)[\sin t - \cos t]R(t_o, t_o) + [\sin t_o - \cos t_o]R(t_o, t_o)} \tag{17}$$

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This problem has multiple singularities occurring at the zeros of the denominator in Eq. (17). The solution can be obtained by successive application of Eqs. (14, 15, and 18)

$$R(t_2, t_o) = W(t_2, t_1) + P(t_2, t_1)[I - R(t_1, t_o)Q(t_2, t_1)]^{-1} R(t_1, t_o) \cdot T(t_2, t_1) \quad (18)$$

over the intervals $[t_o, t_1] = [t_o, n\Delta t]$ and $[t_1, t_2] = [n\Delta t, (n+1)\Delta t]$, $n = 1, 2, \ldots$. [Equation (18) is generally valid for any matrix satisfying the differential Eq. (1), and may also be derived from the semigroup property for fundamental matrices of linear systems.] The values $W(t_3, t_1)$, $T(t_2, t_1)$, $P(t_2, t_1)$, and $Q(t_2, t_1)$ can be obtained by stepwise numerical integration of the initial-value system comprised of Eqs. (1, 2, and 8–13). For the present problem this system has the form

$$\begin{aligned} \dot{W} &= W + W^2 \sin t & W(t_o, t_o) &= 0 \\ \dot{T} &= 0 & T(t_o, t_o) &= 1 \\ \dot{P} &= P & P(t_o, t_o) &= 1 \\ \dot{Q} &= TP \sin t & Q(t_o, t_o) &= 0 \end{aligned}$$

**Table 1 Values of $R(t, t_o)$**

| $t$ | 0 | 0.1 | 0.2 | 0.3 | 0.4 |
|---|---|---|---|---|---|
| 0 | 1.00000 | 1.11110 | 1.24990 | 1.42792 | 1.66394 |
| 0.5 | 1.99066 | 2.47064 | 3.24038 | 4.66548 | 8.16817 |
| 1.0 | <u>29.98016</u> | <u>−19.09487</u> | −7.44914 | −4.72224 | −3.51564 |
| 1.5 | −2.84241 | −2.41874 | −2.13259 | −1.93096 | −1.78568 |
| 2.0 | −1.68051 | −1.60565 | −1.55497 | −1.52471 | −1.51270 |
| 2.5 | −1.51799 | −1.54069 | −1.58197 | −1.64419 | −1.73132 |
| 3.0 | −1.84958 | −2.00870 | −2.22430 | −2.52254 | −2.95035 |
| 3.5 | −3.60051 | −4.68478 | −6.81337 | −12.76006 | <u>−111.5806</u> |
| 4.0 | <u>16.46436</u> | 7.69154 | 5.04654 | 3.78118 | 3.04624 |
| 4.5 | 2.57121 | 2.24318 | 2.00676 | 1.83168 | 1.70008 |

The tabulated values of $R(t, t_o)$ were obtained by the above-mentioned method for the solution of Eq. (16) corresponding to $t_o = 0$ and $R(0, 0) = 1.0$. The functions $W$, $T$, $P$, and $Q$ were computed over increments of length $\Delta t = 0.1$ by a fourth-order Runge-Kutta algorithm. The tabulated values agree with the exact solution to within 1 in the last digit, except for the value at $t = 3.9$ which is in error by 2 places in the last digit. The two singular points within the tabulated range, which occur between the neighboring pairs of underlined points, did not present numerical difficulties. Similar results were obtained for matrix Riccati equations.[10]

In closing, the authors wish to emphasize that they have no computational experience with the method described *as applied to control problems*, and there is no intention to imply that it, or any of the other methods mentioned, will yield practically useful results in all circumstances. Rather, the intent is to point out the existence of more-or-less known techniques for continuing Riccati variables past singularities, so that occurrence of such a singularity need not be interpreted as failure of the associated computational scheme.

## References


[1] Tapley, B. D. and Williamson, W. E., "Comparison of Linear Riccati Equations Used to Solve Optimal Control Problems," *AIAA Journal*, Vol. 10, No. 9, Sept. 1972, pp. 1154–1159.

[2] Polak, E., *Computational Methods in Optimization*, Academic Press, New York, 1971.

[3] Polak, E., "An Historical Survey of Computational Methods in Optimal Control," *SIAM Review*, Vol. 2, No. 2, Pt. 2, April 1973, pp. 553–584.

[4] Denman, E. D., *Coupled Modes in Plasmas, Elastic Media and Parametric Amplifiers*, American Elsevier, Houston, Texas, 1970.

[5] Allen, R. C., Jr. and Wing, G. M., "A Numerical Algorithm Suggested by Problems of Transport in Periodic Media," *Journal of Mathematical Analysis and Applications*, Vol. 29, No. 1, Jan. 1970, pp. 141–157.

[6] Allen, R. C., Jr. and Wing, G. M., "Generalized Trigonometric Identities and Invariant Imbedding," *Journal of Mathematical Analysis and Applications*, Vol. 42, No. 2, May 1973, pp. 397–408.

[7] Allen, R. C., Jr. and Wing, G. M., "An Invariant Imbedding Algorithm for the Solution of Inhomogeneous Two-Point Boundary-Value Problems," *Journal of Computational Physics*, to be published.

[8] Scott, M. R., "An Initial-Value Method for the Eigenvalue Problem for Systems of Ordinary Differential Equations," *Journal of Computational Physics*, Vol. 12, No. 3, July 1973, pp. 334–347.

[9] Casti, J., Kalaba, R., and Scott, M., "A Proposal for the Calculation of Characteristic Functions for Certain Differential and Integral Operators via Initial-Value Methods," *Journal of Mathematical Analysis and Applications*, Vol. 41, No. 1, Jan. 1973, pp. 1–13.

[10] Musial, W., "A Numerical Investigation of the P-Equations Relating the Matrix Riccati Equation to a Linear State Equation," M.S. thesis, 1972, Univ. of Houston, Houston, Texas.